Solutions to Question Sheet 9, Differentiation IV. v1 2019-20

More Examples for extra practice

1. Evaluate
$$\lim_{x\to 0}\frac{\sin\left((3+x)^2\right)-\sin 9}{x}.$$

   Hint: Recognize it as a derivative.

   Solution Set $f(x)=\sin\left((3+x)^2\right)$, then
$$\frac{\sin\left((3+x)^2\right)-\sin 9}{x}=\frac{f(x)-f(0)}{x-0}.$$

   So
$$\lim_{x\to 0}\frac{\sin\left((3+x)^2\right)-\sin 9}{x}=f'(0)\,.$$

   The function $f(x)$ is the composite of $\sin y$ and $y=(3+x)^2$, both of which are differentiable. By the *composite rule* for differentiation we have
$$\begin{aligned}\frac{df(x)}{dx} &= \frac{d\sin y}{dy}\frac{dy}{dx}=\cos y\times 2\,(3+x)\\ &= 2\,(3+x)\cos\left((3+x)^2\right).\end{aligned}$$

   Hence
$$\lim_{x\to 0}\frac{\sin\left((3+x)^2\right)-\sin 9}{x}=2\,(3+x)\cos\left((3+x)^2\right)\Big|_{x=0}=6\cos 9.$$

2. Let $f_n:\mathbb{R}\to\mathbb{R}$ be defined by
$$f_n(x)=\begin{cases} x^n & x>0\\ 0 & x=0\\ -x^n & x<0\end{cases}.$$

By verifying the definition prove that for all $n \geq 1$, the function f_n is $n-1$ times differentiable with $f_n^{(n-1)}$ is continuous on $\mathbb{R}$ but f_n is not n times differentiable.

Solution Proof by induction. Base case $n = 1$. Then $f_1(x) = |x|$ which is known to be continuous but not differentiable on $\mathbb{R}$.

Assume that f_k is $k-1$ times differentiable with $f_k^{(k-1)}$ continuous on $\mathbb{R}$ but not differentiable. Consider f_{k+1}. The only difficulty is checking differentiability at $x = 0$. I leave it to the student to check that both one-sided limits of

$$\frac{f_{k+1}(x) - f_{k+1}(0)}{x - 0}$$

as $x \to 0+$ and $x \to 0-$ are zero so $f_{k+1}^{(1)}(0)$ exists and is 0. It is then easy to show that

$$f_{k+1}^{(1)}(x) = (k+1) f_k(x)$$

for all $x \in \mathbb{R}$. Thus, by the inductive hypothesis, $f_{k+1}^{(1)}$ is $k-1$ times differentiable with $\left(f_{k+1}^{(1)}\right)^{(k-1)}$ continuous on $\mathbb{R}$ but not differentiable. Hence f_{k+1} is k times differentiable with $f_{k+1}^{(k)}$ continuous on $\mathbb{R}$ but not differentiable.

Therefore, by induction, for all $n \geq 1$ we have f_n is $n-1$ times differentiable with $f_n^{(n-1)}$ is continuous on $\mathbb{R}$ but not differentiable.

3. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} x^2 \sin\left(\dfrac{\pi}{x^2}\right) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

   i) Use the definition to show that g is differentiable at $x = 0$ and find the value of $g'(0)$.

   ii) Use the Chain Rule for Differentiation to find $g'(x)$ for all $x \neq 0$.

   iii) Calculate

   $$g'\left(\frac{1}{\sqrt{2n}}\right)$$

   where $n \in \mathbb{N}$.

   iv) Prove that $\lim_{x\to 0} g'(x)$ does not exist and so g' is not continuous.

Aside: In the previous question a second derivative existed, was continuous but not differentiable. In this question, the first derivative existed but was not continuous. By examining the family of functions $x^k \sin\left(\pi/x^\ell\right)$ for integers k and ℓ you can construct functions that have exactly the number of derivatives you want at a point but then its last derivative is either not continuous at that point or, if continuous, not differentiable.

Solution i) The definition of differentiable involves a limit. We will consider the two one-sided limits.

Consider first $x > 0$, when

$$\frac{g(x) - g(0)}{x - 0} = \frac{x^2 \sin\left(\pi/x^2\right) - 0}{x - 0},$$

Thus

$$\lim_{x\to 0+} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0+} x \sin\left(\frac{\pi}{x^2}\right).$$

By the Sandwich Rule this limit exists and equals 0.

Next, for $x < 0$ we have

$$\frac{g(x) - g(0)}{x - 0} = \frac{0 - 0}{x - 0} = 0.$$

Thus

$$\lim_{x\to 0-} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0-} 0 = 0.$$

Since the two-sided limits exist and are equal we deduce that

$$g'(0) = \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0}$$

exists and equals 0.

ii) For non-zero $x > 0$ we simply use the rules of differentiation to get

$$g'(x) = 2x \sin\frac{\pi}{x^2} - \frac{2\pi}{x} \cos\frac{\pi}{x^2}.$$

iii) Substituting $x = 1/\sqrt{2n}$ gives

$$g'\left(\frac{1}{\sqrt{2n}}\right) = \frac{2}{\sqrt{2n}}\sin(2n\pi) - 2\pi\sqrt{2n}\cos(2n\pi) = -2\pi\sqrt{2n}, \qquad (1)$$

since $\sin(2n\pi) = 0$ and $\cos(2n\pi) = 1$ for all $n \in \mathbb{N}$.

iv) Assume that $\lim_{x\to 0} g'(x)$ does exist, with value ℓ say.

Choose $\varepsilon = 1$ in the definition of the limit to find $\delta > 0$ such that $0 < |x| < \delta$ implies $|g'(x) - \ell| < 1$, i.e.

$$\begin{aligned} |g'(x)| &= |g'(x) - \ell + \ell| \\ &\le |g'(x) - \ell| + |\ell| \qquad \text{by triangle inequality} \\ &< 1 + |\ell|. \qquad (2) \end{aligned}$$

But if we choose $n \in \mathbb{N}$ sufficiently large and set $x_n = 1/\sqrt{2n}$ we can have both $0 < |x_n| < \delta$ and, by (1), $|g'(x_n)| = 2\pi\sqrt{2n} > 1 + |\ell|$. This contradicts (2) and thus our assumption, that the limit exists, is false.

4. Using the Mean Value Theorem prove that

$$\arcsin x < \frac{x}{\sqrt{1-x^2}}$$

for all $0 < x < 1$.

Solution Let

$$f(t) = \frac{t}{\sqrt{1-t^2}} - \arcsin t$$

for $0 < t < 1$, noting that $f(0) = 0$. Then by Question 8, Sheet 7,

$$f'(t) = \frac{\sqrt{1-t^2} - \frac{-2t}{2\sqrt{1-t^2}}}{1-t^2} - \frac{1}{\sqrt{1-t^2}} = \frac{t}{(1-t^2)^{3/2}} > 0$$

for our t. The Mean Value theorem applied to f the interval $[0, x]$ implies that there exists $0 < c < x$ for which

$$f(x) - f(0) = f'(c)(x - 0) > 0.$$

This rearranges to the required result.

5. Using the Mean Value Theorem prove that

$$\ln(1+x) > \frac{x}{1+\frac{x}{2}}$$

for $x > 0$.

Solution For $t \geq 0$ define

$$f(t) = \ln(1+t) - \frac{t}{1+\frac{t}{2}}.$$

Given $x > 0$ consider f on the interval $[0, x]$. It satisfies the condition of the Mean Value Theorem and so there exists $0 < c < x$ such that

$$f(x) - f(0) = f'(c)(x-0).$$

Yet

$$\begin{aligned} f'(t) &= \frac{1}{1+t} - \frac{(2+t)\,2 - 2t}{(2+t)^2} = \frac{(2+t)^2 - 4(1+t)}{(1+t)(2+t)^2} \\ &= \frac{t^2}{(1+t)(2+t)^2} > 0 \end{aligned}$$

for all $t > 0$. Thus, since $f(0) = 0$, we have $f(x) = f'(c)\,x > 0$ since $x > 0$. This is the required result.

6. (Exam 2009)

   i) Prove that
   $$2^x = x^2$$
   has at least three real solutions.

   ii) Prove that it has exactly three real solutions.

Solution i) Let $f(x) = 2^x - x^2$ and look for some sign changes. Randomly choosing integer values for x leads to $f(-1) = -3/4$, $f(0) = 1$. We have a sign change so by the Intermediate Value theorem there is a solution between -1 and 0.

Trying more integer values for x we find $f(1) = 1$, $f(2) = 0$ (giving a solution!) $f(3) = -1$ and $f(4) = 0$ giving another solution. Thus we have 3 solutions.

ii) To show that it has exactly three solutions we assume, for a contradiction, that it has more, i.e. at least four. Then by the result in Question 1, Sheet 7, there exists a $c \in \mathbb{R}$ for which $f^{(3)}(c) = 0$. In this case $f^{(3)}(x) = (\ln 2)^3 2^x$ which is never zero. This contradiction means the function has at most 3 solutions. Since we know it has at least 3, we conclude it has exactly 3 solutions.

7. Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Prove that if $a > 0$ there exists $c \in (a, b)$ such that

$$f(b) = f(a) + \ln\left(\frac{b}{a}\right) cf'(c),$$

Solution Apply the Cauchy Mean Value Theorem to f and $g(x) = \ln x$. This is allowable since $x \in [a, b]$ and it is being assumed that $a > 0$. Then there exists $c \in [a, b]$ for which

$$\frac{f(b) - f(a)}{\ln b - \ln a} = \frac{f'(c)}{1/c}.$$

This rearranges to the stated result.

8. i) Prove that

$$\arcsin x + \arccos x$$

is constant on $(-1, 1)$.

What is the value of this constant?

Hint: look at the derivative.

ii) What can you say about

$$\arctan u + \arctan \frac{1}{u}$$

for $u > 0$.

Solution i. From Question 8, Sheet 7, we have

$$\frac{d}{dx}\left(\arcsin x + \arccos x\right) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$

for $x \in (-1, 1)$. Thus $\arcsin x + \arccos x$ is constant on $(-1, 1)$. To find its value take $x = 0$, when $\arcsin 0 + \arccos 0 = \pi/2$. Hence

$$\arcsin x + \arccos x = \frac{\pi}{2}$$

for $x \in (-1, 1)$.

Note this is simply the result that the two non-right angles in a right angled triangle sum to $\pi/2$.

ii. Similarly

$$\frac{d}{du}\left(\arctan u + \arctan \frac{1}{u}\right) = \frac{1}{1+u^2} + \frac{-\frac{1}{u^2}}{1+\frac{1}{u^2}} = \frac{1}{1+u^2} - \frac{1}{1+u^2} = 0.$$

So $\arctan u + \arctan \frac{1}{u}$ is constant. Take $u = 1$ to see that

$$\arctan u + \arctan \frac{1}{u} = 2\arctan 1 = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2},$$

for all $u > 0$.

9. Do **not** use L'Hôpital's Rule to evaluate the following limits i-iv, but instead assume the following results:

$$\lim_{x\to 0} \frac{\cos -1}{x^2} = -\frac{1}{2} \quad \text{and} \quad \lim_{x\to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}.$$

i)

$$\lim_{x\to 0} \frac{x\cos x - \sin x}{x^3},$$

Hint. Write $x\cos x - \sin x = x\cos x - x + x - \sin x$.

ii)

$$\lim_{x\to 0} \frac{\tan x - x}{x^3},$$

iii)

$$\lim_{x\to 0}\frac{\tan x - x}{\tan^3 x},$$

iv)

$$\lim_{x\to 0}\frac{\sin 3x - 3x}{x^3},$$

v)

$$\lim_{x\to 0}\frac{\sin 3x - 3\sin x}{x^3}.$$

Solution i) For $x \neq 0$ but near 0,

$$\begin{aligned}\frac{x\cos x - \sin x}{x^3} &= \frac{x\cos x - x + x - \sin x}{x^3} \\ &= \frac{x\cos x - x}{x^3} + \frac{x - \sin x}{x^3} \\ &= \frac{\cos x - 1}{x^2} + \frac{x - \sin x}{x^3} \\ &\to -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3},\end{aligned}$$

as $x \to 0$, using assumptions in the question.

ii) For $x \neq 0$ but near 0,

$$\begin{aligned}\frac{\tan x - x}{x^3} &= \frac{\sin x - x\cos x}{x^3\cos x} = -\frac{1}{\cos x}\left(\frac{x\cos x - \sin x}{x^3}\right) \\ &\to \frac{1}{1}\times\left(-\frac{1}{3}\right) = \frac{1}{3},\end{aligned}$$

as $x \to 0$, using the result from Part i.

iii)

$$\begin{aligned}\frac{\tan x - x}{\tan^3 x} &= \frac{x^3}{\tan^3 x}\times\frac{\tan x - x}{x^3} = \cos^3 x\left(\frac{x}{\sin x}\right)^3\frac{\tan x - x}{x^3} \\ &\to 1\times 1\times\frac{1}{3} = \frac{1}{3}\end{aligned}$$

as $x \to 0$, using Part ii. along with results from lectures.

iv)

$$\frac{\sin 3x - 3x}{x^3} = 27\frac{\sin 3x - 3x}{(3x)^3} = 27f(3x),$$

where $f(x) = (\sin x - x)/x^3$ when $x \neq 0$. By either L'Hôpital's Rule or Question 5ii, Sheet 8, we know that $\lim_{x\to 0} f(x) = -1/6$.

Hence

$$\lim_{x\to 0}\frac{\sin 3x - 3x}{x^3} = 27\lim_{x\to 0} f(3x) = -\frac{27}{6} = -\frac{9}{2}.$$

Note that we have implicitly used a result on limits of a composite $x \longmapsto 3x \longmapsto f(3x)$.

v) .

$$\begin{aligned}\frac{\sin 3x - 3\sin x}{x^3} &= \frac{\sin 3x - 3x + 3x - 3\sin x}{x^3}\\ &= 27\frac{\sin 3x - 3x}{(3x)^3} - 3\frac{\sin x - x}{x^3}\\ &\to 27\times\left(-\frac{1}{6}\right) - 3\times\left(-\frac{1}{6}\right) = -4,\end{aligned}$$

by Part iii.

10. In Question 14, Sheet 7, you were asked to show that

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0\\ 1 & \text{if } x = 0,\end{cases}$$

is differentiable at $x = 0$.

Write down $f'(x)$ for all $x \in \mathbb{R}$. Calculate $f^{(2)}(0)$.

Hint you may recall that $\lim_{x\to 0}(\sin x - x)/x^3 = -1/6$.

Solution

$$f'(x) = \begin{cases} \dfrac{x\cos x - \sin x}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

For $f^{(2)}(0)$ consider

$$\frac{f'(x) - f'(0)}{x - 0} = \frac{x\cos x - \sin x}{x^3}.$$

This has been seen in the previous question, where it was shown to have limit $-1/3$.

11. Use the Composition Rule for Differentiation to prove

i)

$$\frac{d}{dy}\arcsin\left(\frac{1}{\cosh y}\right) = -\frac{1}{\cosh y}$$

for $y > 0$.

ii)

$$\frac{d}{dy}\left(\arctan\left(\sinh y\right)\right) = \frac{1}{\cosh y}$$

for $y \in \mathbb{R}$.

iii) Can you make up an example for arccos with an appropriate hyperbolic function?

Solution i) From Question 8, Sheet 7,

$$\frac{d}{dy}\arcsin y = \frac{1}{\sqrt{1-y^2}},$$

for $-1 < y < 1$. an earlier question. The Composition Rule then gives

$$\begin{aligned} \frac{d}{dy}\arcsin\left(\frac{1}{\cosh y}\right) &= \frac{1}{\sqrt{1-\left(\frac{1}{\cosh y}\right)^2}} \times \left(-\frac{\sinh y}{\cosh^2 y}\right) \\ &= -\frac{\cosh y}{\sinh y} \times \frac{\sinh y}{\cosh^2 y} = -\frac{1}{\cosh y}. \end{aligned}$$

For the first equality we need $-1 < 1/\cosh y < 1$. But since $\cosh y \geq 1$ with equality at $y = 0$ this means $y \neq 0$. We also take the *positive* square root in $\sqrt{\cosh^2 y - 1} = \sinh y$, so $\sinh y \geq 0$. The combination of $y \neq 0$ and $\sinh y \geq 0$ is $y > 0$. .

ii) Again from Question 8, Sheet 7,

$$\frac{d}{dy}(\arctan y) = \frac{1}{1+y^2}$$

for all $y \in \mathbb{R}$. The Composition Rule then gives

$$\begin{aligned} \frac{d}{dy}(\arctan(\sinh y)) &= \frac{1}{1+(\sinh y)^2} \times \cosh y = \frac{\cosh y}{\cosh^2 y} \\ &= \frac{1}{\cosh y}, \end{aligned}$$

for all $y \in \mathbb{R}$.

iii) From Question 8, Sheet 7,

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$

for any $x \in (-1, 1)$. We could replace x by $1/\cosh y$ as done in part (i), and I leave that to the interested Student.

Alternatively, replace x by $\tanh y$ since we know that $\tanh y \in (-1, 1)$ for all $y \in \mathbb{R}$. Then

$$\begin{aligned} \frac{d}{dy}\arccos(\tanh y) &= -\frac{1}{\sqrt{1-(\tanh y)^2}} \times \frac{1}{\cosh^2 y} = -\cosh y \times \frac{1}{\cosh^2 y} \\ &= -\frac{1}{\cosh y}. \end{aligned}$$

Valid for all $y \in \mathbb{R}$.

12. i) Calculate the first six Taylor Polynomials

$$T_{n,0}(\ln(1+x))\big|_{x=1}, \quad 0 \leq n \leq 5.$$

Calculate the first 6 approximations to $\ln 2$, using these polynomials *with an appropriate choice of x.*

ii) Give the Taylor Series for $\ln(1-x)$ and

$$\ln\left(\frac{1+x}{1-x}\right)$$

about 0, along with their intervals of convergence.

Note: The series for $\ln((1+x)/(1-x))$ is due to Gregory, 1668

iii) Calculate the first 6 approximations to $\ln 2$, using the first six Taylor polynomials

$$T_{n,0}(\ln(1-x)), 0 \le n \le 5,$$

with an appropriate choice of x.

iv) Calculate the first 6 approximations to $\ln 2$, using the first six Taylor polynomials

$$T_{n,0}\left(\ln\left(\frac{1+x}{1-x}\right)\right),$$

$0 \le n \le 5$, *with an appropriate choice of* x.

Solution i) Let $f(x) = \ln(1+x)$. Then

$$\begin{aligned}
f^{(1)}(x) &= (1+x)^{-1}, \quad \text{so } f^{(1)}(0) = 1, \\
f^{(2)}(x) &= -(1+x)^{-2}, \quad \text{so } f^{(2)}(0) = -1, \\
f^{(3)}(x) &= 2(1+x)^{-3}, \quad \text{so } f^{(3)}(0) = 2, \\
f^{(4)}(x) &= -3!\,(1+x)^{-4}, \quad \text{so } f^{(4)}(0) = -3!, \\
f^{(5)}(x) &= 4!\,(1+x)^{-5}, \quad \text{so } f^{(5)}(0) = 4!.
\end{aligned}$$

Thus the first 6 approximations to $\ln(1+x)$, i.e. $T_{n,0}(\ln(1+x))$ for $0 \le n \le 5$, are

$$\begin{aligned}
T_{0,0}(\ln(1+x)) &= 0, \\
T_{1,0}(\ln(1+x)) &= x, \\
T_{2,0}(\ln(1+x)) &= x - \frac{x^2}{2}, \\
T_{3,0}(\ln(1+x)) &= x - \frac{x^2}{2} + \frac{x^3}{3}, \\
T_{4,0}(\ln(1+x)) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}, \\
T_{5,0}(\ln(1+x)) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}.
\end{aligned}$$

Choosing $x = 1$ we get a sequence of approximations to ln2 of

$$1, 0.5,\ 0.8\overline{3},\ 0.58\overline{3},\ 0.78\overline{3},\ 0.61\overline{6},\ ....$$

This sequence converges **very** slowly.

ii) From above we see that for each $n \geq 1$, $f^{(n)}(x) = (n-1)!\,(1+x)^{-n}$, so $f^{(n)}(0) = (n-1)!$. Thus the Taylor series for $\ln(1+x)$ is

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + ...$$

which converges for $-1 < x \leq 1$.

Replace x by $-x$ in the Taylor series for $\ln(1+x)$ to get

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - ...\ ,$$

valid for $-1 \leq x < 1$. Note that

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)\,.$$

We would like to obtain the Taylor series for $g(x) = \ln((1+x)/(1-x))$ by subtracting that for $\ln(1-x)$ from the one for $\ln(1+x)$. But you need to justify the subtraction of *infinite* series. To calculate the Taylor series for g we need to calculate $g^{(n)}$ for all $n \geq 1$. But $g(x) = \ln(1+x) - \ln(1-x) = f(x) - h\,(x)$, say, so $g^{(n)}$ can be found as the difference of the derivatives of f and h or, in other words, the n^{th}-term for $\ln((1+x)/(1-x))$ is the difference of the n^{th}-terms for f and h. So we **are** allowed to subtract term-by-term to get

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + ...$$

for $-1 < x < 1$.

iii) Put $x = 1/2$ in $\ln(1-x)$ to get approximations to $\ln 2$ of

$$0.5,\ 0.625,\ 0.\overline{6},\ 0.68229...,\ 0.68854...,\ 0.6911458....,\ ...$$

iv) Put $x = 1/3$ in $\ln((1+x)/(1-x))$ to get approximations to $\ln 2$ of

$$0.\overline{6},\ 0.69135...,\ 0.69300...,\ 0.69313...,\ 0.693146...,\ 0.693147....,...$$

Note $\ln 2 = 0.69315...$ and in each case above we are getting sequences that converge quicker than in the preceding case.

13. What is the maximum *possible* error in using $T_{5,0}f(x)$ to approximate $f(x) = \sin x$ on the interval $[-0.25, 0.25]$?

What is the *actual* error when using the Taylor polynomial to approximate $\sin(12°)$?

Solution There is no need to calculate the Taylor polynomial for $\sin x$, just Lagrange's form of the error. So with $f(x) = \sin x$ we have $f^{(6)}(x) = -\sin x$ and

$$R_{5,0}f(x) = -\frac{\sin c}{6!}x^6$$

for some c between 0 and x. But $|\sin c| \leq 1$ and so, with $|x| \leq 0.25$ we find

$$|R_{5,0}f(x)| \leq \frac{(0.25)^6}{6!} = 3.390844... \times 10^{-7}. \qquad (3)$$

To find the actual error we do need the Taylor polynomial

$$T_{5,0}f(x) = x - \frac{x^3}{6} + \frac{x^5}{120}.$$

The value at 12° or $\pi/15$, is

$$\begin{aligned} T_{5,0}f\left(\frac{\pi}{15}\right) &= \frac{\pi}{15} - \frac{1}{6}\left(\frac{\pi}{15}\right)^3 + \frac{1}{120}\left(\frac{\pi}{15}\right)^5 \\ &\approx 0.2079116943... . \end{aligned}$$

The difference between the value of the Taylor polynomial and the true value of $\sin(\pi/15)$ is $\approx 3.505219... \times 10^{-9}$, smaller, which was to be expected, than the bound in (3).

14. Approximate $f(x) = \sqrt[3]{x}$ by the quadratic $T_{2,8}f(x)$.

How accurate is the approximation when $7 \le x \le 9$?

Solution If $f(x) = x^{1/3}$ then

$$\begin{aligned} f^{(1)}(x) &= x^{-2/3}/3, \\ f^{(2)}(x) &= -2x^{-5/3}/9, \\ f^{(3)}(x) &= 10x^{-8/3}/27. \end{aligned}$$

When $a = 8$, then $f(8) = 2$, $f^{(1)}(8) = 1/12$, and $f^{(2)}(8) = -1/144$, so

$$T_{2,8}f(x) = 2 + \frac{(x-8)}{12} - \frac{(x-8)^2}{288}.$$

The error, in Lagrange's form, is

$$R_{2,8}f(x) = \frac{f^{(3)}(c)}{3!}(x-8)^3$$

for some c between 8 and x. We are told to restrict to $x \in [7, 9]$.

If $x > 8$ then $R_{2,8}f(x) > 0$ but also $8 < c < x < 9$ and so

$$R_{2,8}f(x) = \frac{10\,(x-8)^3}{27 \times 3!c^{8/3}} < \frac{10}{27 \times 3! \times 8^{8/3}} = \frac{10}{27 \times 3! \times 2^8} < 0.000241127.$$

If $x < 8$ then $R_{2,8}f(x) < 0$ but also $7 < x < c < 8$ and so

$$R_{2,8}f(x) = \frac{10\,(x-8)^3}{27 \times 3!c^{8/3}} > -\frac{10}{27 \times 3! \times 7^{8/3}} > -0.000344263.$$

15. Show that the Taylor series for $g(x) = (1+x)^{1/2}$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n\,(2n)!}{4^n\,(1-2n)\,(n!)^2} x^n.$$

Hint You need to show that

$$g^{(n)}(0) = (-1)^{n-1}\,\frac{(2n)!}{4^n n!\,(2n-1)}$$

for all $n \geq 1$.

Solution If $g(x) = (1+x)^{1/2}$ then

$$\begin{aligned} g^{(1)}(x) &= \frac{1}{2}(1+x)^{-1/2} \\ g^{(2)}(x) &= \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-3/2} \\ g^{(3)}(x) &= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1+x)^{-5/2} \\ &\vdots \end{aligned}$$

In general

$$\begin{aligned} g^{(n)}(0) &= \frac{(1-0)}{2}\left(\frac{1-2}{2}\right)\left(\frac{1-4}{2}\right)...\left(\frac{1-2(n-1)}{2}\right) \\ &= (-1)^{n-1}\frac{(2n-3)(2n-5)...1}{2^n} \\ &= (-1)^{n-1}\frac{(2n-3)(2n-4)(2n-5)(2n-6)...2\times 1}{2^n(2n-4)(2n-6)...2} \\ &= (-1)^{n-1}\frac{(2n-3)!}{2^n 2^{n-2}(n-2)(n-3)...1} \\ &= (-1)^{n-1}\frac{(2n-3)!}{4^{n-1}(n-2)!} \\ &= (-1)^{n-1}\frac{1}{4^{n-1}}\frac{n(n-1)}{n!}\frac{(2n)!}{(2n)(2n-1)(2n-2)} \\ &= (-1)^{n-1}\frac{(2n)!}{4^n n!(2n-1)}. \end{aligned}$$

Hence the Taylor Polynomial for $\sqrt{1+x}$ is around $x=0$ is

$$\sum_{n=0}^{\infty}\frac{(-1)^{n-1}(2n)!}{(2n-1)(n!)(4^n)}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{(1-2n)(n!)^2(4^n)}x^n.$$

16. Show that

i) the Taylor series for $f(x) = 1/\sqrt{(1+x)}$ around $x = 0$ is

$$\sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{4^n (n!)^2} x^n,$$

(**Hint** Try to reuse work you have already done. Note that appears in the solution of Question as $2g^{(1)}(x)$, with $g(x) = \sqrt{1+x}$.)

ii) the Taylor series for $h(x) = 1/\sqrt{(1-x^2)}$ around $x = 0$ is

$$\sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} y^{2n}.$$

(**Hint** Use the fact that if the Taylor series of $f(x)$ is $\sum_{n=0}^{\infty} a_n x^n$ then the Taylor series of $f\left(\alpha x^k\right)$ is $\sum_{n=0}^{\infty} a_n \left(\alpha x^k\right)^n$.)

iii) the Taylor Series for $\arcsin x$ around $x = 0$ is

$$\sum_{\ell=0}^{\infty} \frac{(2\ell)! x^{2\ell+1}}{4^{\ell} (2\ell+1) (\ell!)^2}.$$

(**Note** I am **not** asking for you to prove that any of these series converge to the given function but you might want to think about how you could do this.)

Solution i) By the hint given $f(x) = 2g^{(1)}(x)$ in which case, from looking back at the earlier question,

$$\begin{aligned} f^{(n)}(0) &= 2g^{(n+1)}(0) = 2(-1)^n \frac{(2(n+1))!}{4^{n+1}(n+1)!(2(n+1)-1)} \\ &= 2(-1)^n \frac{2(n+1)(2n+1)(2n)!}{4^{n+1}(n+1)\,n!\,(2n+1)} \\ &= (-1)^n \frac{(2n)!}{4^n n!}. \end{aligned}$$

Then the Taylor Series for f is

$$\sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{4^n n!} \frac{x^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{4^n (n!)^2} x^n.$$

ii) With $f(x)$ as in part i, we have that

$$\frac{1}{\sqrt{1-y^2}} = f\left(-y^2\right) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{4^n (n!)^2} \left(-y^2\right)^n = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} y^{2n}.$$

(Of course, y is simply a label and can be replaced by x).

iii) We have seen earlier that on $(-1,1)$ we have

$$\frac{d}{dy} \arcsin y = \frac{1}{\sqrt{1-y^2}}.$$

So if $k(x) = \arcsin x$ and $h(x) = 1/\sqrt{1-x^2}$ then, $k^{(n)}(0) = h^{(n-1)}(0)$. Note the Taylor Series for $h(y)$ is

$$\sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} y^{2n} = \sum_{n=0}^{\infty} \frac{((2n)!)^2}{4^n (n!)^2} \frac{y^{2n}}{(2n)!}.$$

From this we see that

$$h^{(m)}(0) = \frac{((2n)!)^2}{4^n (n!)^2}$$

if $m = 2n$, i.e. m is even, 0 otherwise. Therefore $k^{(n)}(0) = 0$ if n even, while if $n = 2\ell + 1$, then

$$k^{(n)}(0) = h^{(n-1)}(0) = \frac{((2\ell)!)^2}{4^n (\ell!)^2}.$$

Thus the Taylor Series of $k(x) = \arcsin x$ is

$$\sum_{\ell=0}^{\infty} \frac{((2\ell)!)^2}{4^n (\ell!)^2} \frac{x^{2\ell+1}}{(2\ell+1)!} = \sum_{\ell=0}^{\infty} \frac{(2\ell)!}{4^n (\ell!)^2 (2\ell+1)} x^{2\ell+1}.$$

17. Let $f(x) = \sin x$.

i) Prove that

$$f^{(n)}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(\cos\left(n\frac{\pi}{2}\right) + \sin\left(n\frac{\pi}{2}\right)\right)$$

for all $n \geq 1$.

ii) Show that for all $n \geq 1$ both sides of the identity,

$$\cos\left(n\frac{\pi}{2}\right) + \sin\left(n\frac{\pi}{2}\right) = (-1)^{n(n-1)/2} \tag{4}$$

are the same.

Hint: Any n can be written as $n = 4m+r$, where r, the remainder on dividing by 4, takes only the values $r = 0, 1, 2$ or 3. Show that the values of both sides of (4) depend only on r, and so there are only 4 cases to check.

iii) Deduce that the Taylor series for $\sin x$ around $a = \pi/4$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^{n(n-1)/2}}{\sqrt{2}n!}\left(x - \frac{\pi}{4}\right)^n.$$

Prove that this series converges to $\sin x$ for all x.

Solution i) Take $f(x) = \sin x$ and $a = \pi/4$. Then student to check that

$$f^{(n)}(x) = \sin\left(x + n\frac{\pi}{2}\right)$$

and

$$\begin{aligned} f^{(n)}\left(\frac{\pi}{4}\right) &= \sin\left(\frac{\pi}{4} + n\frac{\pi}{2}\right) \\ &= \frac{1}{\sqrt{2}}\left(\sin\left(n\frac{\pi}{2}\right) + \cos\left(n\frac{\pi}{2}\right)\right), \end{aligned}$$

by the addition formula for sine.

ii) We split into two cases. First consider n even, so $n = 2m$. Then

$$(-1)^{\frac{n(n-1)}{2}} = (-1)^{m(2m-1)} = \left((-1)^{2m-1}\right)^m = (-1)^m$$

since $2m-1$ is odd in which case $(-1)^{2m-1} = (-1)$. But also

$$\begin{aligned}
\sin\left(n\frac{\pi}{2}\right) + \cos\left(n\frac{\pi}{2}\right) &= \sin(m\pi) + \cos(m\pi) \\
&= 0 + (-1)^m \\
&= (-1)^{\frac{n(n-1)}{2}}.
\end{aligned}$$

In the second case consider n odd, so $n = 2m+1$. Then

$$(-1)^{\frac{n(n-1)}{2}} = (-1)^{m(2m+1)} = \left((-1)^{2m+1}\right)^m = (-1)^m.$$

And

$$\begin{aligned}
\sin\left(n\frac{\pi}{2}\right) + \cos\left(n\frac{\pi}{2}\right) &= \sin\left(m\pi + \frac{\pi}{2}\right) + \cos\left(m\pi + \frac{\pi}{2}\right) \\
&= (-1)^m + 0 \\
&= (-1)^{\frac{n(n-1)}{2}}.
\end{aligned}$$

Hence, by combining both cases,

$$\sin\left(n\frac{\pi}{2}\right) + \cos\left(n\frac{\pi}{2}\right) = (-1)^{\frac{n(n-1)}{2}}$$

for all $n \in \mathbb{N}$.

iii) Combining Parts i and ii gives

$$f^{(n)}\left(\frac{\pi}{4}\right) = (-1)^{\frac{n(n-1)}{2}} / \sqrt{2}$$

for all n. Hence

$$\sum_{n=0}^{\infty} \frac{(-1)^{\frac{n(n-1)}{2}}}{\sqrt{2}n!}\left(x - \frac{\pi}{4}\right)^n.$$

We next have to show that this series converges to $\sin x$ for all $x \in \mathbb{R}$.

Let $x \in \mathbb{R}$ be given. Then, for some c between $\pi/4$ and x,

$$\begin{aligned}
\left|R_{n,\frac{\pi}{4}}(\sin x)\right| &= \left|\frac{f^{(n+1)}(c)}{(n+1)!}\left(x - \frac{\pi}{4}\right)^n\right| \\
&\leq \frac{1}{(n+1)!}\left|x - \frac{\pi}{4}\right|^{n+1} \to 0
\end{aligned}$$

as $n \to \infty$ since $\left\{|x - \pi/4|^{n+1} / (n+1)!\right\}_{n\geq 1}$ is a null sequence for all x.